EXISTENCE AND UNIQUENESS OF THE SOLUTION OF THE ELASTIC-PLASTIC TORSION PROBLEM FOR A CYLINDRICAL BAR OF OVAL CROSS-SECTION (*)

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The problem of determining the stresses arising during the torsion of a cylindrical bar of elastic-ideal plastic material was posed a long time ago [1 and 2]. The major difficulty is that the boundary separating the elastic and plastic regions is unknown and its determination is part of the processes of finding the solution. The exact solution for a circular cross-section was found in [1], and that for an almost elliptic cross-section was found in [3]. When the cross-section is a polygon [4], this problem is reduced to the determination of two functions of a complex variable that are analytic in the upper half-plane and satisfy certain boundary-value conditions on the real axes. In [3 and 5], an inverse method was proposed in which the shape of the elastic core is used to find the cross-section. In [6], the inverse method was used to reduce the elastic-plastic torsion problem to a nonlinear singular integral equation which was not investigated any further. In [7], necessary conditions for the existence of the solution of the elastic-plastic torsion problem were ascertained. Various approximate methods were suggested in [4 and 8 to 12].

Below we will study the case of an oval cross-section for angles of twist such that the elastic-plastic boundary has no point in common with the longitudinal surface of the bar. By means of a Legendre transformation, the present problem has been reduced to the Dirichlet problem in a circle for the Monge-Ampère equation of the elliptic type. Moreover, the elastic-plastic boundary is determined from the normal derivative of the solution on the boundary of the circle. The existence and uniqueness of the solution of the elastic-plastic torsion problem has been proved. A number of estimates of practical interest have also been obtained.

1. Formulation of problem. We will consider the elastic-plastic torsion of a cylindrical bar whose cross-section F is bounded by the strictly concave contour Γ. We will assume that the radius of curvature $\rho(s) > 0$ exists at each point of the contour Γ and that as a function of the

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arc-length s it is a function of class C^2, i.e. that $d^2\rho(s)/ds^2$ is continuous. It is clear that $\infty > \rho_{max} \geqslant \rho(s) \geqslant \rho_{min} > 0$, where ρ_{max}, and ρ_{min} are the maximum and minimum values of the radius of curvature, respectively. We will use the triangular Cartesian axes xyz with the z-axis parallel to the generatrix of the cylindrical surface of the origin O in F. Let G be the shear modulus, k the plastic constant, α the angle of twist per unit length, and Ω the area of F. We will use the notation $a = 2^{-1}G^{-1}\alpha^{-1}k$. The twisting has a clockwise sense when viewed in the positive direction of the z-axis. We will assume that there is an elastic core, namely the simply-connected region D with boundary L lying entirely inside Γ (Fig.1). The doubly-connected region bounded by Γ and L will be denoted by B. In this region the material is in a completely plastic state. The indices x, y will be used to indicate partial derivatives, as well as for the usual purpose of indicating the components of stress, where this does not lead to confusion.

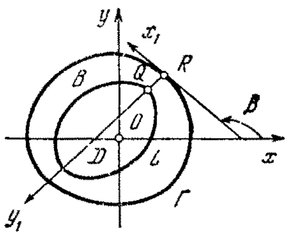

Fig. 1

The elastic-plastic torsion problem, which henceforth will be called problem A, is formulated as follows.

P r o b l e m A. Given a simply-connected region F bounded by the oval Γ satisfying the above-mentioned smoothness conditions. In the simply-connected region D, which together with its boundary L lies inside F, it is required to find to within an arbitrary additive constant the function $\psi(x,y)$ that is (1) single-valued and continuous in $F + \Gamma$, (2) has continuous first order partial derivatives ψ_x, ψ_y in $F + \Gamma$, (3) has continuous second order partial derivatives ψ_{xx}, ψ_{xy}, ψ_{yy} in D, and (4) satisfies the following conditions:

$$\psi_{xx} + \psi_{yy} = -a^{-1}, \quad \psi_x^2 + \psi_y^2 < 1 \qquad \text{in region } D \tag{1.1}$$

$$\psi_x^2 + \psi_y^2 = 1 \ (B = F - (D + L)) \qquad \text{in region } L + B + \Gamma \tag{1.2}$$

$$\operatorname{grad}\psi = \mathbf{n}_\Gamma \qquad \text{on boundary } \Gamma \tag{1.3}$$

where $\mathbf{n}_\Gamma$ is the unit outward normal to the boundary curve Γ.

N o t e 1.1. The quantity $a = 2^{-1}G^{-1}\alpha^{-1}k$ is regarded as parameter $0 < \alpha < \infty$.

N o t e 1.2. It follows from (1.3) that the function $\psi(x,y)$ is constant on Γ. We will assume that

$$\psi(x, y)_\Gamma = 0 \tag{1.4}$$

N o t e 1.3. The shear stresses are

$$\tau_{xz} = k\psi_y, \qquad \tau_{yz} = -k\psi_x$$

N o t e 1.4. The problem of the flow of a dilatant fluid through a tube [13] of elliptic cross-section F has a similar formulation. Another hydrodynamic interpretation of Problem A was given by von Mises [5].

2. Uniqueness of solution. Let R be an arbitrary point on the contour Γ. Let x_1Ry_1 be the moving coordinate system (Fig.1) in the xy-plane formed by the tangent Rx_1 to Γ at R pointed in the direction of increasing arc-length, and the inward normal Ry_1. We will assume that the tangent to Γ at the point of intersection with the x-axis is perpendicular to the x-axis. The angle between Rx_1 and Ox will be denoted by β. The equation of the oval Γ can be represented in the form [14]

$$x^\circ(\beta) = \frac{dM(\beta)}{d\beta}\cos\beta + M(\beta)\sin\beta, \qquad y^\circ(\beta) = \frac{dM(\beta)}{d\beta}\sin\beta - M(\beta)\cos\beta \qquad (^1/_2\pi \leqslant \beta < {^5/_2}\pi) \tag{2.1}$$

where $M(\beta)$ is the basic function on Γ. We have the relation

$$\rho(\beta) = M(\beta) + \frac{d^2M(\beta)}{d\beta^2} \tag{2,2}$$

where $\rho(\beta)$ is the radius of curvature of Γ as a function of β.

Note 2.1. According to (2.1), each $\beta \in [^1/_2\pi, {^5/_2}\pi)$ is associated with one and only one point $R \in \Gamma$.

Note 2.2. It is clear that $M(\beta) \in C^4$, i.e. $d^4M(\beta)\,/\,d\beta^4$ is a continuous function for $\beta \in [^1/_2\pi, {^5/_2}\pi)$.

Definition. The curve L situated within Γ has property E if the following conditions are fulfilled:

a) The curve L admits a representation of the form

$$X(\beta) = x^\circ(\beta) - N(\beta)\sin\beta, \quad Y(\beta) = y^\circ(\beta) + N(\beta)\cos\beta \qquad (^1/_2\pi \leqslant \beta < {^5/_2}\pi) \tag{2.3}$$

where $x^\circ(\beta)$ and $y^\circ(\beta)$ are taken from (2.1) and $N(\beta)$ has period 2π and is a continuous function of β in $[^1/_2\pi, {^5/_2}\pi)$, with $0 < N(\beta) < \rho(\beta)$.

b) For two distinct arbitrary angles $\beta = \beta_1$ and $\beta = \beta_2$ in $[\frac{1}{2}\pi, \frac{5}{2}\pi]$ the line segments R_1Q_1 and R_2Q_2 do not have a point of intersection (*). Here, each point $R_i \in \Gamma$ corresponds to a β_i satisfying (2.1) and each point $Q_i \in L$ corresponds to a β_i satisfying (2.3) ($i = 1, 2$).

Note 2.3. It is clear that L is a simple Jordan curve such that to each value of $\beta \in [^1/_2\pi, {^5/_2}\pi)$ there is one and only one point $Q \in L$; given by (2.3). Moreover, $N(\beta)$ is the ordinate of this point in the system of coordinates x_1Ry_1, i.e it is the length of the segment QR (Fig.1).

Theorem 2.1. If a solution of Problem A having property E on the contour L exists, and if the function $\psi(x,y)$ is twice continuously differentiable (**) in $L + B + \Gamma$, then this solution is unique.

Let it be assumed that Problem A has a solution with the properties mentioned in Theorem 2.1; then the following assertions are valid.

2.1°. In the $xy\psi$-space the functions $\psi(x,y)$, where the points (x,y) lie in $L + B + \Gamma$, generate a surface whose parametric representation is

$$x = \frac{dM(\beta)}{d\beta}\cos\beta + [M(\beta) - u]\sin\beta \qquad (^1/_2\pi \leqslant \beta < {^5/_2}\pi) \tag{2.4}$$

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*) In particular, points Q_1 and Q_2 do not coincide.

**) i.e. $\psi(x,y)$ has continuous second order derivatives which are continuously extensible from B onto L and Γ.

$$y = \frac{dM(\beta)}{d\beta}\sin\beta - [M(\beta) - u]\cos\beta \qquad (0 \leqslant u \leqslant N(\beta)) \tag{2.4 cont.}$$

$$\psi = u$$

It is to be noted that Formulas (2.4) also result if one uses the Cauchy method to solve the Cauchy problem for Equation (1.2) using the conditions (1.3) and (1.4), where Γ is taken in the form (2.1). The validity of assertion 2.1° follows [15] from the fact that $\psi(x,y)$ has continuous second order derivatives in $L + B + \Gamma$ and that the projections on the xy-plane of the characteristics of the integral surface satisfying Equation (1.2) and condition (1.3) do not intersect (*) in $L + B + \Gamma$, since they coincide with the normals to Γ.

Corollary 2.1. In $L + B + \Gamma$ the following relations hold:

$$\psi_x = -\sin\beta, \qquad \psi_y = \cos\beta, \qquad \psi - x\psi_x - y\psi_y = M(\beta) \tag{2.5}$$

Note 2.4. From Note 2.1 and relations (2.5) which hold in particular on L, it follows that each of the functions $\psi_x(x,y)$, $\psi_y(x,y)$ do not assume on L any particular value more than twice.

2.2°. Everywhere in region D

$$\psi_{xx}\psi_{yy} - \psi_{xy}^2 \neq 0 \tag{2.6}$$

Proof. Let point $(x_0, y_0) \in D$ and

$$\psi_{xx}\psi_{yy} - \psi_{xy}^2 = 0$$

Without limiting the generality, one can assume (**) that

$$\psi_{xy}(x_0, y_0) = \psi_{xx}(x_0, y_0) = 0$$

Then, ([16], page 428) the harmonic function $\psi_x(x,y)$ assumes the value $\psi_x(x_0, y_0)$ at at least four distinct points on the contour (***).

2.3°. Formulas

$$\xi = -\psi_x(x, y), \qquad \eta = -\psi_y(x, y), \qquad (x, y) \in D + L \tag{2.7}$$

effect a homeomorphic mapping of $D + L$ onto the circle $K + C$ in the $\xi\eta$-plane. This circle is defined by the inequality $\xi^2 + \eta^2 \leqslant 1$.

This assertion is vindicated as follows: (1) since the functions $\psi_x(x,y)$ and $\psi_y(x,y)$ are continuous in $D + L$, Formulas (2.7) represent a one-to-one mapping of L onto the circumference C of unit radius, i.e. $\xi^2 + \eta^2 = 1$; (2) the Jacobian of mapping (2.7) is nonzero in region D ([17], page 26).

2.4°. Everywhere in region D

$$\psi_{xx}\psi_{yy} - \psi_{xy}^2 > 0$$

Proof. Let the contrary be assumed. Then the index of the isolated singular point in the continuous vector field $\mathbf{q} = (\psi_x, \psi_y)$ will be -1. This, however, is impossible, since the index of L relative to $\mathbf{q}$ is 1 (see, for example, [18]).

Corollary 2.2. Everywhere in D

$$\psi_{xx} < 0, \qquad \psi_{yy} < 0$$

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*) Thus, property E of curve Γ guarantees the single-valued continuation from Γ onto L of the Cauchy data (1.3) along the projections of the characteristics onto the xy-plane.

**) By rotating the coordinate system, the second mixed partial derivative can always be made zero at point (x_0, y_0). At the same time, the Laplacian and the Hessian remain constant, and the property mentioned in Note 2.4 also remains unaltered.

***) If at the point (x_0, y_0) all partial derivatives with respect to x and y of ψ_x are zero, then ψ_x is constant in the vicinity of point (x_0, y_0) and consequently also in $D + L$.

2.5°. Let us introduce the notation

$$\Phi = -x\psi_x - y\psi_y + \psi \tag{2.8}$$

and regard Φ as a function of the variables ξ, η defined in (2.7). The function $\Phi(\xi, \eta)$ is then continuous in the closed circle $K + C$, and in K it satisfies (*) Equations

$$\Phi_{\xi\xi}\Phi_{\eta\eta} - \Phi_{\xi\eta}^2 - a(\Phi_{\xi\xi} + \Phi_{\eta\eta}) = 0 \tag{2.9}$$

and the inequalities

$$\Phi_{\xi\xi} > 0, \qquad \Phi_{\eta\eta} > 0 \tag{2.10}$$

On the circumference C this function satisfies the condition

$$\Phi_\xi(\xi, \eta)|_C = M(\theta + {}^1/_2\pi) \tag{2.11}$$

where θ is the polar angle in the $\xi\eta$-plane. In circle K we have the identities

$$\Phi_\xi(\xi, \eta) = x, \qquad \Phi_\eta(\xi, \eta) = y \tag{2.12}$$

The proof of 2.5° follows from the properties of the Legendre transformation [19].

2.6°. Let us use the notation

$$w(\xi, \eta) = \Phi(\xi, \eta) - {}^1/_2 a(\xi^2 + \eta^2) + {}^1/_2 a \tag{2.13}$$

The function $w(\xi, \eta)$ is continuous in $K + C$, and on the circumference of the circle its value is

$$w(\xi, \eta)|_C = M(\theta + {}^1/_2\pi) \tag{2.14}$$

and inside K it satisfies the Monge-Ampère equation

$$w_{\xi\xi}w_{\eta\eta} - w_{\xi\eta}{}^2 = a^2 \tag{2.15}$$

and the inequalities

$$w_{\xi\xi} > 0, \qquad w_{\eta\eta} > 0 \tag{2.16}$$

N o t e 2.5. There exists at most one function $w(\xi, \eta)$ that (1) is continuous in $K + C$, and (2) satisfies Equation (2.15), inequalities (2.16) in K, and condition (2.14) on C.

P r o o f o f t h e o r e m 2.1. First of all we remark that, by virtue of 2.1° and the uniqueness of the Dirichlet problem, there can not be two distinct solutions of Problem A for Equation (1.1) with the same curves L having property E. Moreover, by virtue of Note 2.5 the function $\Phi(\xi,\eta)$, which satisfies Equation (2.9) and inequality (2.10) in K and condition (2.11) on C, is unique. Finally, by virtue of Formula (2.12) the region D and consequently curve L are uniquely determined by function $\Phi(\xi, \eta)$. This proves Theorem 2.1.

C o r o l l a r y 2.3. If

$$\alpha < kG^{-1}\Omega^{-1/2}\pi^{1/2} \tag{2.17}$$

Problem A has no solution possessing the properties mentioned in Theorem 2.1.

P r o o f . Let L_δ denote the mapping under (2.7) of the circle C_δ, given by $\xi^2 + \eta^2 = 1 - \delta$, $0 < \delta < 1$, and let D_δ denote the closed region bounded by L_δ. When α satisfies inequality (2.17), this contradicts the following easily established inequality:

$$\pi(1 - \delta) = \frac{1}{2}\int_{L_\delta} \psi_x d\psi_y - \psi_y d\psi_x = \iint_{D_\delta} (\psi_{xx}\psi_{yy} - \psi_{xy}{}^2)\, dxdy < \frac{1}{4a^2}\Omega$$

3. Existence of solution. T h e o r e m 3.1. If

$$\alpha > k / G\rho_{\min} \tag{3.1}$$

*) Here and in what follows the subscripts ξ, η indicate partial differentiation.

then the solution of Problem A exists. Moreover, the contour L has property E and the function $\psi(x,y)$ has continuous second order derivatives in $L + B + \Gamma$.

The proof of this theorem consists in constructing the contour L and the function $\psi(x,y)$ satisfying the conditions formulated in Section 1. For this we will adopt the line of argument in Section 2.

3.1°. There exists a function $w(\xi,\eta)$ which has continuous second order derivatives in the closed circle $K + C$ defined by the inequality $\xi^2 + \eta^2 \leqslant 1$, and which satisfies Equation (2.15) and inequalities (2.16) in $K + C$ and condition (2.14) on the unit circle C.

The proof of this assertion follows from [20 to 22] if it is assumed that

$$M(\theta + {}^1/_2\pi) \in C^4 \qquad 0 \leqslant \theta < 2\pi$$

Note 3.1. For the radial derivative of function $w(\xi,\eta)$ on C the following inequality holds $(0 \leqslant \theta \leqslant 2\pi)$:

$$w_r(1, \theta) + \frac{d^2M(\theta + {}^1/_2\pi)}{d\theta^2} \geqslant m_0 > 0, \quad w_r(1, \theta) = \frac{\partial w(r, \theta)}{\partial r}\bigg|_{r=1}, \quad r = \sqrt{\xi^2 + \eta^2} \tag{3.2}$$

where m_0 is a constant depending on the maximum modulus of the derivatives of order up to and including the fourth of $M(\theta + \frac{1}{2}\pi)$ ([16], page 136).

Note 3.2. Let (ξ_1, η_1) and (ξ_2, η_2) be two distinct points on the closed curve $K + C$. Then we have the inequality [23]

$$(\xi_2 - \xi_1)[w_\xi(\xi_2, \eta_2) - w_\xi(\xi_1, \eta_1)] + (\eta_2 - \eta_1)[w_\eta(\xi_2, \eta_2) - w_\eta(\xi_1, \eta_1)] > 0 \tag{3.3}$$

Note 3.2°. Having proved the existence of function $w(\xi,\eta)$ in 3.1°, let us define function $\Phi(\xi,\eta)$ by means of Equation (2.13), when (ξ,η) lies in $K + C$. Then, in the circle $K + C$, function $\Phi(\xi,\eta)$ satisfies Equation (2.9) and inequality (2.10); on C it satisfies condition (2.11). Thus, for any two distinct points (ξ_1, η_1) and (ξ_2, η_2) in $K + C$ the following inequality is satisfied

$$a^2[(\xi_2 - \xi_1)^2 + (\eta_2 - \eta_1)^2] < [\Phi_\xi(\xi_2, \eta_2) - \Phi_\xi(\xi_1, \eta_1)]^2 + [\Phi_\eta(\xi_2, \eta_2) - \Phi_\eta(\xi_1, \eta_1)]^2 \tag{3.4}$$

3.3°. Now we introduce the notation

$$x = \Phi_\xi(\xi, \eta), \qquad y = \Phi_\eta(\xi, \eta) \tag{3.5}$$

where $(\xi, \eta) \in K + C$ and function $\Phi(\xi,\eta)$ is the same as that in 3.2°. Then Formula (3.5) furnishes a homeomorphic mapping of $K + C$ onto a certain (closed) region $D + L$ of the xy-plane. Equation L of the image of C (with the notation $\beta = \theta + \frac{1}{2}\pi$) can be written in the form (2.3), where

$$N(\beta) = M(\beta) - \Phi_r(1, \beta - {}^1/_2\pi)$$

$$\Phi_r(1, \theta) = \left(\frac{\partial \Phi}{\partial r}\right)_{r=1} \qquad ({}^1/_2\pi \leqslant \beta < {}^5/_2\pi) \tag{3.6}$$

Proof. The homeomorphic character follows from (3.4) if it is assumed that $\Phi_\xi(\xi, \eta)$ and $\Phi_\eta(\xi, \eta)$ are continuous in $K + C$. Setting $\xi = \cos\theta$, $\eta = \sin\theta$ and expressing $\Phi_\xi(\xi, \eta)|_C$ and $\Phi_\eta(\xi, \eta)|_C$ in terms of $\Phi_r(1, \theta)$ and $\Phi_\theta(1, \theta) = dM(\theta + {}^1/_2\pi)/d\theta$, we obtain the equation of L in the form (2.3) with $N(\beta)$ defined by Equation (3.6).

Note 3.3. To each $\beta \in [{}^1/_2\pi, {}^5/_2\pi)$ corresponds one and only one point on the curve L. The curve L is smooth.

Corollary 3.1. The vector

$$\mathbf{v}_L(\beta_0) = (dY(\beta_0)/d\beta, -dX(\beta_0)/d\beta) \tag{3.7}$$

is collinear with the vector normal to L at the point β_0, $\beta_0 \in [{}^1/_2\pi, {}^5/_2\pi)$.

Proof. As θ increases, the circumference C is traversed in the positive sense (region K remains to the left) and the Jacobian of the mapping (3.5) is strictly positive on L by virtue of (2.9) and (2.10). Thus, the curve L is also traversed in the positive sense when $\beta = \theta + \frac{1}{2}\pi$ increases. Whence follows that the vector

$$t_L(\beta_0) = (dX(\beta_0)/d\beta, \quad dY(\beta_0)/d\beta)$$

is directed along the tangent to L in the direction of positive rotation. The vector $\nu_L(\beta_0)$ can then be obtained by rotation of $t_L(\beta_0)$ clockwise through an angle $\frac{1}{2}\pi$

3.4°. We will write

$$\psi = -\xi\Phi_\xi - \eta\Phi_\eta + \Phi \tag{3.8}$$

and consider ψ as a function of x and y determined by Formula (3.5). Then the function $\psi(x,y)$ has a continuous derivative in $D + L$, satisfies Equation (1.1) in region D, and fulfills relations (2.5) and (2.6) on L. Thus, for · ~ Equations $-\psi_x(x, y) = \xi$ and $-\psi_y(x, y) = \eta$ hold.

Corollary 3.2. The angle between the vector $\tau(\beta) = (\sin\beta, -\cos\beta)$ and the vector $\nu_L(\beta)$ (see (3.7)) is strictly less than $\frac{1}{2}\pi$ for arbitrary $\beta \in [{}^1/_2\pi, {}^5/_2\pi)$.

Indeed, by virtue of (3.2) the scalar product of these vectors is positive definite.

3.5°. Let Q_1 and Q_2 be two distinct points on curve L which correspond, respectively, to the distinct points $\beta_1, \beta_2 \in [{}^1/_2\pi, {}^5/_2\pi)$. Then, the half-lines $l(Q_1)$ and $l(Q_2)$ that originate in points Q_1 and Q_2 and have the direction of the vectors $\tau(\beta_1) = (\sin\beta_1 - \cos\beta_1)$ and $\tau(\beta_2) = (\sin\beta_2, -\cos\beta_2)$, respectively, will not intersect.

Proof. The equations of the half-lines $l(Q_1)$ and $l(Q_2)$ can be written as

$$x_i(\lambda_i) = X(\beta_i) + \lambda_i \sin\beta_i, \qquad y_i(\lambda_i) = Y(\beta_i) - \lambda_i \cos\beta_i$$

$$0 \leqslant \lambda_i < \infty \qquad (i = 1, 2)$$

Let us assume that the half-lines $l(Q_1)$ and $l(Q_2)$ intersect; this means that there are values λ_1^*, λ_2^* such that $\lambda_1^* + \lambda_2^* > 0$ and such that

$$x_1(\lambda_1^*) = x_2(\lambda_2^*), \; y_1(\lambda_1^*) = y_2(\lambda_2^*)$$

Now we will make use of inequality (3.3) in assuming

$$\xi_i = \sin\beta_i, \; \eta_i = -\cos\beta_i, \; w_\xi(\xi_i, \eta_i) = X(\beta_i) - a\xi_i, \; w_\eta(\xi_i, \eta_i) = Y(\beta_i) - a\eta_i \; (i = 1, 2)$$

It will then be found that

$$-(\lambda_1^* + \lambda_2^*)[1 - \cos(\beta_1 - \beta_2)] > 0$$

which is impossible.

3.6°. For an arbitrary $\beta \in [{}^1/_2\pi, {}^5/_2\pi)$ the function $N(\beta)$ defined by Formula (3.6) is positive definite.

Proof. Let $P_0 = (\cos\theta_0, \sin\theta_0)$ be an arbitrary fixed point on the circumference C. We will introduce the function

$$\Phi^*(\xi, \eta) = dM(\theta_0 + {}^1/_2\pi)/d\theta \, r\sin(\theta - \theta_0) + [M(\theta_0 + {}^1/_2\pi) - \rho_{\min}]\cos(\theta - \theta_0) +$$

$$+ar^2 + \rho_{\min} - a$$

and the notation

$$\Delta(\xi, \eta) = \Phi(\xi, \eta) - \Phi^*(\xi, \eta), \qquad \Delta(\xi, \eta)|_{r=1} = g(\theta), \qquad (\xi, \eta) \in K + C$$

where the function $\Phi(\xi,\eta)$ is the same as in 3.2°. Since

$$g(\theta_0) = dg(\theta_0) \,|\, d\theta = 0, \qquad d^2g(\theta) \,|\, d\theta^2 + g(\theta) \geqslant 0$$

it follows that $g(\theta) \geqslant 0$ in the whole interval $0 \leqslant \theta < 2\pi$. In so far as the function $\Delta(\xi,\eta)$ satisfies in K the elliptic differential equation

$$\Phi_{\eta\eta}\Delta_{\xi\xi} - 2\Phi_{\xi\eta}\Delta_{\xi\eta} + \Phi_{\xi\xi}\Delta_{\eta\eta} = 0$$

it follows that $\Delta(\xi, \eta) \geqslant 0$ in region $K + C$. Consequently

$$\frac{\partial\Delta(\xi,\eta)}{\partial r}\bigg|_{P_0'}\leqslant 0$$

i.e. $N(\beta_0)\geqslant\rho_{\min}-2a$, where $\beta_0=\theta_0+{}^1/{}_2\pi$. Taking (3.1) into account, we find that $N(\beta_0)$ is positive.

3.7°. Let Q be an arbitrary point on curve L corresponding to some $\beta\in[{}^1/{}_2\pi, {}^5/{}_2\pi)$, and $l(Q)$ be the half-line originating in point Q and directed parallel to the vector

$$\tau(\beta)=(\sin\beta, -\cos\beta)$$

If the point R is chosen on $l(Q)$ such that the distance of point Q from point R is equal to $N(\beta)$, where $N(\beta)$ is defined by (3.6), then the totality of points R corresponding to all $\beta\in[{}^1/{}_2\pi, {}^5/{}_2\pi)$, constitutes the curve Γ. Thus β is the angle of inclination to the x-axis of the tangent at point R to curve Γ, where the direction of the tangent is that of positive motion around Γ.

3.8°. Curve L has property E.

We must obviously check that $N(\beta)<\rho(\beta)$ for $\beta\in[{}^1/{}_2\pi, {}^5/{}_2\pi)$. This follows from inequality (3.2).

3.9°. Let B denote the region between Γ and L. We will define the function $\psi(x,y)$ in $L+B+\Gamma$ by means of Equations (2.4) when

$${}^1/{}_2\pi\leqslant\beta<{}^5/{}_2\pi,\qquad 0\leqslant u\leqslant N(\beta)$$

where $N(\beta)$ is determined by (3.6). Then, in $L+B+\Gamma$ function $\psi(x,y)$ satisfies Equation (1.3) and on Γ the conditions (1.3) and (1.4). Equations (2.5) are satisfied on L.

3.10°. The contour L determined by Equations (2.3) with $N(\beta)$ in the form (3.6) and function $\psi(x,y)$ defined in $F+\Gamma$ by the method mentioned in 3.4° and 3.9°, is the solution of Problem A. This completes the proof of Theorem 3.1.

C o r o l l a r y 3.3. For arbitrary $\beta\in[{}^1/{}_2\pi, {}^5/{}_2\pi)$ we have

$$\rho_{\min}-kG^{-1}\alpha^{-1}\leqslant N(\beta)<\rho(\beta)-k2^{-1}G^{-1}\alpha^{-1}\tag{3.9}$$

C o r o l l a r y 3.4. If

$$\alpha\leqslant k2^{-1}G^{-1}\rho_{\min}{}^{-1}$$

then no solution of Problem A having the properties mentioned in Theorem 2.1 exists.

4. Some properties of the solution of Problem A. 4.1°. Let $\alpha_2>\alpha_1$. We will denote the two values of α introduced in Sections 1 to 3 by α_1 and α_2, respectively. Then for an arbitrary $\beta\in[{}^1/{}_2\pi, {}^5/{}_2\pi)$ we have

$$N_1(\beta)+\frac{1}{2}\frac{k}{G}\left(\frac{1}{\alpha_1}-\frac{1}{\alpha_2}\right)\leqslant N_2(\beta)\leqslant N_1(\beta)+\frac{k}{G}\left(\frac{1}{\alpha_1}-\frac{1}{\alpha_2}\right)\tag{4.1}$$

P r o o f . We will use the notation

$$A'={}^1/{}_2(w_{1\eta\eta}+w_{2\eta\eta}),\qquad B'={}^1/{}_2(w_{1\xi\eta}+w_{2\xi\eta}),\qquad C'={}^1/{}_2(w_{1\xi\xi}+w_{2\xi\xi})$$

$$H(\xi,\eta)=w_2(\xi,\eta)-w_1(\xi,\eta),\qquad (\xi,\eta)\in K+C,\qquad \xi^2+\eta^2\leqslant 1$$

$$H^*(\xi,\eta)=w_1(\xi,\eta)-w_2(\xi,\eta)-\frac{k}{4G}\left(\frac{1}{\alpha_*}-\frac{1}{\alpha_2}\right)(\xi^2+\eta^2-1),\qquad \alpha_*<\alpha_1$$

Since the following inequalities hold on curve K

$$A'>0,\qquad B'>0,\qquad A'C'-(B')^2>0$$

$$A'H_{\xi\xi}-2B'H_{\xi\eta}+C'H_{\eta\eta}<0,\qquad A'H_{\xi\xi}{}^*-2B'H_{\xi\eta}{}^*+C'H_{\eta\eta}{}^*<0$$

the functions $H(\xi,\eta)$ and $H^*(\xi,\eta)$ can not attain their respective minima in K [22]. Provided $H(\xi,\eta)|_C=H^*(\xi,\eta)|_C=0$, we have $\partial H(1,\theta)/\partial r\leqslant 0$ and $\partial H^*(1,\theta)/\partial r\leqslant 0$. Whence follows the validity of inequality (4.1), if one takes a value α_* slightly different from α_1.

4.2°. If the curve Γ is symmetric with respect to some axis, then curve L is also symmetric with respect to the same exis.

P r o o f . Let Γ be symmetric with respect to the x-axis, i.e.

$$M(\theta + {}^1/{}_2\pi) = M(-\theta + {}^1/{}_2\pi)$$

and let the function $\Phi(\xi, \eta)$, $(\xi, \eta) \in K + C$, have the same sense as in Section 3. By virtue of the Note 2.5, $\Phi_r(1, -\theta) = \Phi_r(1, \theta)$.

N o t e 4.1. It is known that it is impossible to construct in $F + \Gamma$ a continuous solution of Equation (1.2) with boundary condition (1.3) such that the derivatives ψ_x and ψ_y are continuous in $F + \Gamma$ (because the index of Γ relative to a continuous vector field $q = (\psi_x, \psi_y)$ in $F + \Gamma$ is equal to unity [18]). However, such a construction is possible in $F + \Gamma - l$, where l is a certain region lying in F. We will limit consideration to certain contours Γ for which l consists of straight-line segments, and will prove that, for an arbitrary value of α, l lies inside region D.

a) The curve Γ is symmetric, is elongated along the x-axis, and has only four vertices (the vertex of an oval is a point where the curvature is extremal; every oval has at least four vertices). In this case, l is the segment of the x-axis [24] joining the centers of curvature S_1, S_2 of the contour at points T_1 and T_2 lying on the x-axis (see Fig.2). By virtue of (3.9), L intersects the x-axis at a point lying between T_1 and S_1 and also at a point between T_2 and S_2. If L touches or intersects the segment S_1S_2, then it follows by virtue of 4.2° that region D would not be simply connected. Thus, l lies inside D.

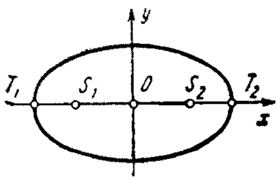

Fig. 2

b) Let the convex curve Γ_n approximate to a regular polygon and be defined by Equation [25]

$$x = R\,[f\cos t + (n-1)^{-1} f^{1-n}\cos(n-1)\,t]$$
$$y = R\,[f\sin t - (n-1)^{-1} f^{1-n}\sin(n-1)\,t]$$
$$0 \leqslant t < 2\pi, \qquad f = (n-1)^{1/n} + \varepsilon, \qquad \varepsilon > 0$$

In this case curve l consists of n segments joining the points of intersection of each of the symmetry axes of Γ_n with the centers of curvature of the points on Γ_n corresponding to $t = 2\pi i/n$ $(i = 0, 1, \ldots, (n-1))$. By virtue of (3.9) and 4.2°, l lies inside L.

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